

# Minimal symmetric Darlington synthesis

L. Baratchart\*, P. Enqvist†, A. Gombani‡, M. Olivi\*

May 3, 2011

**Abstract.** We consider the symmetric Darlington synthesis of a  $p \times p$  rational symmetric Schur function  $S$  with the constraint that the extension is of size  $2p \times 2p$ . Under the assumption that  $S$  is strictly contractive in at least one point of the imaginary axis, we determine the minimal McMillan degree of the extension. In particular, we show that it is generically given by the number of zeros of odd multiplicity of  $I_p - SS^*$ . A constructive characterization of all such extensions is provided in terms of a symmetric realization of  $S$  and of the outer spectral factor of  $I_p - SS^*$ . The authors's motivation for the problem stems from Surface Acoustic Wave filters where physical constraints on the electro-acoustic scattering matrix naturally raise this mathematical issue.

**Keywords.** *symmetric Darlington synthesis, inner extension, MacMillan degree, Riccati equation, symmetric Potapov factorization.*

## 1 Introduction

The Darlington synthesis problem has a long history which goes back to the time when computers were not available and the synthesis of non-lossless circuits was a hard problem: the brilliance of the Darlington synthesis was that it reduced any synthesis problem to a lossless one. In mathematical terms, given a  $(p \times p)$  Schur function  $S$ , say, in the right half-plane, the problem is to imbed  $S$  into a  $(m + p) \times (m + p)$ -inner function  $\mathcal{S}$  so that:

$$\mathcal{S} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S \end{pmatrix}, \quad \mathcal{S}(i\omega)\mathcal{S}^*(i\omega) = I_{m+p}, \quad \omega \in \mathbb{R}. \quad (1)$$

This problem was first studied by Darlington in the case of a scalar rational  $S$  [D1], was generalized to the matrix case [B1], and finally carried over to non-rational  $S$  [A2, D2]. We refer the reader to the nice surveys [BM, D3] for further references and generalizations (*e.g.* to the non-stationary case). An imbedding of the form (1) will be called a *Darlington synthesis* or *inner extension*, or even sometimes a *lossless extension* of  $S$ .

A Darlington synthesis exists provided that  $S(i\omega)$  has constant rank a.e., that the determinant of  $I_p - S(i\omega)S^*(i\omega)$  (viewed as an operator on its range) satisfies the Szegő condition (see *e.g.* [G]), and that  $S$  is pseudo-continuable across the imaginary axis meaning that there is a meromorphic function in the left half-plane whose nontangential limits on  $i\mathbb{R}$  agree with  $S(i\omega)$  a.e. [A2, A3, D2,

---

\*Laurent Baratchart and Martine Olivi are with INRIA, BP 93, 06902 Sophia-Antipolis Cedex, FRANCE, {Laurent.Baratchart@sophia.inria.fr}, {Martine.Olivi@sophia.inria.fr}

†Per Enqvist is with the Division of Optimization and Systems Theory, KTH, Stockholm, Sweden {penqvist@math.kth.se}

‡Andrea Gombani is with ISIB-CNR, Corso Stati Uniti 4, 35127, Padova, Italy, {gombani@isib.cnr.it}

D3, DH]. If moreover  $S$  has the conjugate-symmetry, then  $\mathcal{S}$  may be chosen with this property. When  $S$  is rational the above conditions are fulfilled so that a Darlington synthesis always exists. In addition, it can be carried-out without augmenting the McMillan degree (*i.e.* we may require in (1) that  $\deg \mathcal{S} = \deg S$ ) and choosing  $m = p$ ; this follows easily from Fuhrmann's realization theory [F1] and the arguments in [D3] or else from more direct computations carried out in [AV, GR]. In particular  $\mathcal{S}$  can be chosen rational, and also to have real coefficients if  $S$  does.

When  $S$  is the scattering matrix of an electric  $p$ -pole without gyrators [B1], the reciprocity law entails that  $S$  is symmetric and the question arises whether the extension  $\mathcal{S}$  can also be made symmetric; this would result in a Darlington synthesis which is itself free from gyrators. In [AV] it is shown that a symmetric Darlington synthesis of a symmetric rational  $S$  indeed exists and, although one can no longer preserve the degree while keeping  $m = p$ , he can at least ensure that  $\deg \mathcal{S} \leq 2\deg S$ . The existence of a symmetric Darlington synthesis for non-rational functions has been studied in [A3], in the slightly different but equivalent setting of  $J$ -inner extensions.

In [AV] it is also shown that, by increasing the *size*  $m$  to  $p + n$ , where  $n$  is the degree of  $S$ , it is possible to construct a *symmetric* extension of exact degree  $n$ . However, such an increase of  $m$  is not always appropriate. In fact, although the original motivations from circuit synthesis that brought the problem of lossless imbedding to the fore are mostly forgotten today, the authors of the present paper were led to raise the above issue in connection with the modeling of Surface Acoustic Waves filters [BEGO]. In this context, physical constraints impose  $m = p$ , so that each block of the electro-acoustic scattering matrix  $\mathcal{S}$  in (1) has to be of size  $p \times p$ .

It is thus natural to ask the following : *given a symmetric rational  $S$ , what is the minimal degree of a symmetric lossless extension  $\mathcal{S}$ ?* This is the problem that we consider. For *scalar* systems, this minimal degree has been known for a while and can be found, for instance in [YWP] (see also Section 3 below). The present paper will generalize this to the matrix-valued case. We restrict our attention to the case where  $S$  is strictly contractive in at least one point of the imaginary axis. This implies that the extension will have size  $2p$ . For the general case, that is, with extensions of lower size, the analysis seems to be more difficult and it will possibly be treated in a subsequent paper.

In Section 2 we introduce some notations. In Section 3, we shed light on the problem by discussing the elementary *scalar case*, that is,  $p = 1$ . In Section 4 we recall some results of Gohberg and Rubinstein [GR] about a state space construction of an inner extension preserving the degree and we characterize all inner extensions in terms of minimal ones. In Section 5 we present a simple method to construct (possibly unstable) symmetric unitary extensions. In Section 6 we finally produce a symmetric inner extension of minimal degree. In Section 7 we discuss the symmetric *and conjugate symmetric* unitary extension of a rational symmetric Schur function which is conjugate symmetric (*i.e.* that has real coefficients), and we show on an example that its minimal degree is generally *larger* than the one attainable without the conjugate-symmetry requirement.

## 2 Preliminaries and notations

Throughout, if  $M$  is a complex matrix, we let  $\text{Tr}(M)$  stand for its trace,  $M^T$  for its transpose, and  $M^*$  for its transpose-conjugate. We denote respectively by

$$\Pi^+ = \{s \in \mathbb{C}; \text{Re } s > 0\} \quad \text{and} \quad \Pi^- = \{s \in \mathbb{C}; \text{Re } s < 0\}$$

the right and left half-planes.

In System Theory, a rational function whose poles lie in  $\Pi^-$  is called *stable*, and a rational function which is finite (resp. vanishing) at infinity is called *proper* (resp. *strictly proper*). System Theory is often concerned with functions having the *conjugate symmetry*:  $W(\bar{s}) = \overline{W(s)}$ , but we shall not make this restriction unless otherwise stated. A rational function has the conjugate-symmetry if, and only if, it has real coefficients.

For  $W(s)$  a matrix-valued function on  $\mathbb{C}$ , we define its para-hermitian conjugate  $W^*$  to be:

$$W^*(s) := W(-\bar{s})^*. \quad (2)$$

Note that  $*$  has two different meanings depending on its position with respect to the variable; this slight ambiguity is common in the literature and allows for a simpler notation.

Note that  $W^*(i\omega) = W(i\omega)^*$  on the imaginary axis, and if  $W$  is a polynomial then  $W^*$  is also a polynomial whose zeros are reflected from those of  $W$  across the imaginary axis.

We say that a rational  $p \times m$  matrix-valued function  $S$  holomorphic on  $\Pi^+$  is a *Schur function* if it is contractive:

$$S(s)S(s)^* \leq I_p, \quad s \in \Pi^+. \quad (3)$$

A rational  $p \times p$  Schur function  $S$  is said to be *lossless*, or *inner*, if

$$S(i\omega)S(i\omega)^* = I_p, \quad \omega \in \mathbb{R}. \quad (4)$$

The scalar rational inner functions are of the form  $q^*/q$  where  $q$  is a polynomial whose roots lie in  $\Pi^-$ ; if  $\deg q = n$ , such a function is called a *Blaschke product of degree  $n$* . A (normalized) Blaschke product of degree 1 is just a Möbius transform of the type

$$b_\xi(s) := (s - \xi)/(s + \bar{\xi}), \quad \xi \in \Pi^+. \quad (5)$$

The natural extension to the matrix case is given by

$$B_\xi(s) = \begin{pmatrix} b_\xi(s) & 0 \\ 0 & I_{p-1} \end{pmatrix}, \quad \xi \in \Pi^+, \quad (6)$$

which is the most elementary example of an inner function of degree 1. Actually, it is a result of Potapov [P, D4] that these and unitary matrices together generate all rational inner matrices. More precisely, if  $Q$  is such a matrix and  $\xi_1, \dots, \xi_n$  its zeros (*i.e.* the zeros of its determinant which is a Blaschke product) ordered arbitrarily counting multiplicities, there exist complex unitary matrices  $U_1, \dots, U_{n+1}$  such that

$$Q = U_1 B_{\xi_1} U_2 B_{\xi_2} \dots U_n B_{\xi_n} U_{n+1}. \quad (7)$$

An inner matrix like  $U_1 B_{\xi_1} U_2$  is often called an *elementary Blaschke factor*.

Given a proper rational matrix  $S$ , we shall write

$$S = \left( \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) \quad (8)$$

whenever  $(A, B, C, D)$  is a *realization* of  $S$ , in other words whenever  $S(s) = C(sI_n - A)^{-1}B + D$  where  $A, B, C, D$  are complex matrices of appropriate sizes. Because  $S$  in this case is the so-called transfer function [AV, BGK, KFA] of the linear dynamical system:

$$\dot{x} = Ax + Bu, \quad y = Cx + Du, \quad (9)$$

with state  $x$ , input  $u$ , and output  $y$ , we say sometimes that  $A$  is a *dynamics matrix* for  $S$ . The matrices  $C$  and  $B$  are respectively called the *output* and *input* matrices of the realization. Every proper rational matrix has infinitely many realizations, and a realization is called *minimal* if  $A$  has minimal size. This minimal size will be taken as definition of the *McMillan degree* of  $S$ , abbreviated as  $\deg S$ . As is well-known [KFA, BGK], the realization (8) is minimal if and only if *Kalman's criterion* is satisfied, that is if the two matrices:

$$\begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}, \quad \begin{bmatrix} C^T & A^T C^T & \dots & (A^T)^{n-1} C^T \end{bmatrix}, \quad (10)$$

are surjective, where  $n$  denotes the size of  $A$ .

The surjectivity of the first matrix expresses the *reachability* of the system, and that of the second matrix its *observability*. Any two minimal realizations can be deduced from each other by a linear change of coordinates:

$$(A, B, C, D) \mapsto (TAT^{-1}, TB, CT^{-1}, D), \quad T \text{ an invertible matrix,}$$

so that a dynamics matrix of minimal size for  $S$  is well-defined up to similarity. In particular the eigenvalues of  $A$  depend only on  $S$  and they are in fact its poles, the *multiplicity* of a pole being its total multiplicity as an eigenvalue by definition. The sizes of the Jordan blocks associated to an eigenvalue are called the *partial multiplicities* of that eigenvalue. The partial multiplicities may be computed as follows. Performing elementary row and column operations on  $S$ , one can put it in *local Smith form* at  $\xi$  (see *e.g.* [GLR, sec.7.2.] or [BO2, BGR]):

$$S(s) = E(s) \text{diag}[(s - \xi)^{\nu_1}, (s - \xi)^{\nu_2}, \dots, (s - \xi)^{\nu_k}, 0, \dots, 0] F(s) \quad (11)$$

where  $E(s)$  and  $F(s)$  are rational matrix functions that are finite and invertible at  $\xi$  while  $k$  is the rank of  $S$  as a rational matrix and  $\nu_1 \leq \nu_2 \leq \dots \leq \nu_k$  are relative integers. These integers are uniquely determined by  $S$  and sometimes called its *partial multiplicities* at  $\xi$ . Note that  $\xi$  is a *pole* if, and only if, there is at least one negative partial multiplicity at  $\xi$ . In fact, the negative partial multiplicities at  $\xi$  are precisely the partial multiplicities of  $\xi$  as a pole of  $S$ .

One says that  $\xi$  is a *zero* of  $S$  if the local Smith forms exhibits at least one positive partial multiplicity at  $\xi$ , and the positive partial multiplicities at  $\xi$  are by definition the partial multiplicities of  $\xi$  as a zero. If  $S$  is invertible as a rational matrix, it is clear from (11) that the poles of  $S^{-1}$  are the zeros of  $S$ , with corresponding partial multiplicities. Note also that a zero may well be at the same time a pole, which causes many of the difficulties in the analysis of matrix valued functions. When  $S$  is inner, which is our main concern here, this does not happen because its poles lie in  $\Pi^-$  and its zeros in  $\Pi^+$ .

A rational matrix has real coefficients if, and only if, there exists a minimal realization which is real, *i.e.* such that  $A$ ,  $B$ ,  $C$ , and  $D$  are real matrices. As is customary in System Theory, we occasionally refer to a proper rational matrix as being a *transfer function*. If it happens to have the conjugate-symmetry, we say it is a *real transfer function*.

The system-theoretic interpretation (9) of (8) makes it easy to compute a realization for a product of transfer-functions. In fact, if  $S_1$  is  $m \times k$  and  $S_2$  is  $k \times p$ , and if

$$S_1 = \left( \begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array} \right), \quad S_2 = \left( \begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array} \right), \quad (12)$$

then a short computation shows that the following two realizations hold:

$$S_2 S_1 = \left( \begin{array}{cc|c} A_1 & 0 & B_1 \\ B_2 C_1 & A_2 & B_2 D_1 \\ \hline D_2 C_1 & C_2 & D_2 D_1 \end{array} \right), \quad S_2 S_1 = \left( \begin{array}{cc|c} A_2 & B_2 C_1 & B_2 D_1 \\ 0 & A_1 & B_1 \\ \hline C_2 & D_2 C_1 & D_2 D_1 \end{array} \right). \quad (13)$$

Likewise, if  $k = m$  and  $D_1$  is invertible (so that  $S_1$  is *a fortiori* invertible as a rational matrix), then

$$S_1^{-1} = \left( \begin{array}{cc|c} A_1 - B_1 D_1^{-1} C_1 & B_1 D_1^{-1} & \\ \hline -D_1^{-1} C_1 & D_1^{-1} & \end{array} \right). \quad (14)$$

Using (10), it is immediate that the minimality of (12) implies that of (14). In contrast, the realizations (13) need not be minimal even if the realizations (12) are: pole-zero cancellations may occur in the product  $S_2 S_1$  to the effect that the multiplicity of a pole may not be the sum of its multiplicity as a pole of  $S_2$  (*i.e.* an eigenvalue of  $A_2$ ) and as pole of  $S_1$  (*i.e.* an eigenvalues of  $A_1$ ). One instance where (13) is minimal occurs when  $S_1, S_2$  have full rank and no zero of  $S_1$  is a pole of  $S_2$  and no zero of  $S_2$  is a pole of  $S_1$  (see [CN]). This, in particular, is satisfied when  $S_1$  and  $S_2$  are inner, implying that the McMillan degree of a product of inner functions is the sum of the McMillan degrees. By (7), this in turn implies that the McMillan degree of an inner function is also the degree of its determinant viewed as a scalar Blaschke product.

Whenever  $S$  is a transfer function, its transpose  $S^T$  clearly has the same McMillan degree as  $S$ . A square transfer function  $S$  is called *symmetric* if  $S = S^T$ , and then a realization is called *symmetric* if  $A = A^T, B^T = C$  and  $D = D^T$ . It is not too difficult to see that a transfer function is symmetric if, and only if, it has a minimal realization which is symmetric [FH]. The latter *may be complex* even if  $S$  is a real transfer function.

### 3 The case of a scalar Schur function.

For getting an idea of the solution to our problem, we first consider the symmetric inner extension of a scalar rational Schur function to a  $2 \times 2$  inner rational function, that is we assume momentarily that  $p = m = 1$ . This case has been considered in [YWP]. Put

$$S = \frac{p_1}{q},$$

where  $p_1$  and  $q$  are coprime polynomials such that  $\deg\{p_1\} \leq \deg\{q\}$ ,  $p_1$  is not identically zero,  $|p_1(i\omega)| \leq |q(i\omega)|$  for  $\omega \in \mathbb{R}$ , and  $q$  has roots in the open left half-plane only. The McMillan degree of  $S$  is just the degree of  $q$  in this case. As the orthogonal space to a nonzero vector  $v = (a, b)^T \in \mathbb{C}^2$  is spanned by  $(-\bar{b}, \bar{a})^T$ , it is easily checked that every rational inner extension  $\mathcal{S}$  of  $S$ , when all its entries are written over a common denominator, say,  $dq$  where  $d$  is a stable monic polynomial, is of the form

$$\mathcal{S} = \frac{1}{dq} \begin{bmatrix} e^{i\theta_1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} (dp_1)^* & -p_2^* \\ p_2 & dp_1 \end{bmatrix} \begin{bmatrix} e^{i\theta_2} & 0 \\ 0 & 1 \end{bmatrix}$$

where  $\theta_1, \theta_2 \in \mathbb{R}$  and  $p_2$  is a polynomial solution of degree at most  $\deg\{dq\}$  to the spectral factorization problem:

$$dp_1(dp_1)^* + p_2 p_2^* = dq(dq)^*, \quad (15)$$

whose solvability is ensured by the contractivity of  $S$ . Clearly the extension is symmetric if and only if  $-e^{i\theta_1}p_2^* = e^{i\theta_2}p_1$ , which is compatible with (15) if, and only if, all zeros of the polynomial

$$dq(dq)^* - dp_1(dp_1)^* = dd^*(qq^* - p_1p_1^*)$$

have even multiplicity. Consider the polynomial

$$\mu := qq^* - p_1p_1^*, \quad (16)$$

and single out its roots of even multiplicity by writing  $\mu = (r_1r_1^*)^2r_2r_2^*$ , where  $r_1$  and  $r_2$  are stable coprime polynomials and all the roots of  $r_2$  are simple. For  $dd^*\mu$  to have roots of even multiplicity only, it is then necessary that  $r_2$  divides  $d$ . Therefore, as the McMillan degree of an inner function is the degree of its determinant, we get

$$\deg\{\mathcal{S}\} = \deg\{dq\} \geq \deg\{r_2q\}. \quad (17)$$

On another hand, a symmetric inner extension of McMillan degree  $\deg\{r_2q\}$  is explicitly given by

$$\mathcal{S}_m = \begin{bmatrix} -\frac{p_1^*}{q} \frac{r_2^*}{r_2} & \frac{r_1r_1^*r_2^*}{q} \\ \frac{r_1r_1^*r_2^*}{q} & \frac{p_1}{q} \end{bmatrix} = \begin{bmatrix} -\frac{p_1^*}{q} & \frac{r_1r_1^*r_2^*}{q} \\ \frac{r_1r_1^*r_2^*}{q} & \frac{p_1}{q} \end{bmatrix} \begin{bmatrix} \frac{r_2^*}{r_2} & 0 \\ 0 & 1 \end{bmatrix}. \quad (18)$$

Thus we see already in the scalar case that the minimal attainable degree for a symmetric inner extension of  $S$  is the degree of  $S$  augmented by half the number of zeros of  $\mu$  of odd multiplicity. Formulas (17) and (18) should be compared with the corresponding formulas (98) and (95) in [YWP].

As  $1 - SS^* = \mu/(qq^*)$ , the zeros of  $\mu$  are the zeros of  $1 - SS^*$  augmented by the common zeros to  $p_1$  and  $q^*$  and the common zeros to  $p_1^*$  and  $q$ ; the latter of course are reflected from the former across the imaginary axis, counting multiplicities. In particular a degree-preserving symmetric Darlington synthesis requires special conditions that can be rephrased as:

- (i) the zeros of  $1 - SS^*$  have even multiplicities,
- (ii) each common zero to  $S$  and  $(S^*)^{-1}$ , if any, is common with even multiplicity.

**Remark 3.1** *Note that (i) is automatically fulfilled for those zeros located on the imaginary axis, if any, so the condition really bears on the non-purely imaginary zeros. Note also that (ii) concerns those zeros of  $S$ , if any, whose reflection across the imaginary axis is a pole of  $S$ ; by the coprimeness of  $p_1$  and  $q$ , such zeros are never purely imaginary.*

Our goal is to generalize the previous result to matrix-valued contractive rational functions.

## 4 Inner extensions.

We shall restrict our study to the case where the function  $S$  to be imbedded is strictly contractive at infinity:  $\|S(\infty)\| < 1$ . If  $S$  is strictly contractive at some finite point  $i\omega_0$ , the change of variable  $s \rightarrow 1/(s - i\omega_0)$  will make it contractive at infinity and such a transformation preserves rationality and the McMillan degree while mapping  $\Pi^+$  onto itself, hence our results carry over immediately

to this case. But if  $S$  is strictly contractive *at no point* of the imaginary axis, then our method of proof runs into difficulties and the answer to the minimal degree symmetric inner extension issue will remain open. To recap, we pose the following problem:

*Given a  $p \times p$  symmetric rational Schur function which is strictly contractive at infinity, what is the minimal McMillan degree of a  $2p \times 2p$  inner extension  $\mathcal{S}$  of  $S$  which is also symmetric :*

$$\mathcal{S} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S \end{pmatrix}, \quad S_{11} = S_{11}^T, \quad S_{21} = S_{12}^T, \quad \mathcal{S}(i\omega)\mathcal{S}^*(i\omega) = I_{2p}, \quad \omega \in \mathbb{R}. \quad (19)$$

#### 4.1 Inner extensions of the same McMillan degree.

Our point of departure will be the solution to the Darlington synthesis problem for a rational function in terms of realizations. Let  $S$  be a Schur  $p \times p$  function which is strictly contractive at infinity and let

$$S = \left( \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) \quad (20)$$

be a minimal realization of  $S$  of degree  $n$ . The strict contractivity at infinity means that  $I_p - D^*D$  and  $I_p - DD^*$  are positive definite. Therefore we may set

$$\hat{A} = A + BD^*(I_p - DD^*)^{-1}C, \quad (21)$$

$$\hat{B} = B(I_p - D^*D)^{-1/2}, \quad (22)$$

$$\hat{C} = (I_p - DD^*)^{-1/2}C, \quad (23)$$

and subsequently we define:

$$H = \begin{bmatrix} -\hat{A}^* & -\hat{C}^*\hat{C} \\ \hat{B}\hat{B}^* & \hat{A} \end{bmatrix}. \quad (24)$$

**Lemma 1** *Assuming  $S$  is a Schur function strictly contractive at infinity given by (20), then the matrix  $H$  defined in (24) is a dynamics matrix of  $(I_p - SS^*)^{-1}$ . Furthermore  $H$  is Hamiltonian, i.e.*

$$H^* \begin{bmatrix} 0 & I_p \\ -I_p & 0 \end{bmatrix} = - \begin{bmatrix} 0 & I_p \\ -I_p & 0 \end{bmatrix} H.$$

**Proof.** By definition

$$S^* = \left( \begin{array}{c|c} -A^* & -C^* \\ \hline B^* & D^* \end{array} \right),$$

then, from (13)

$$I_p - SS^* = \left( \begin{array}{cc|c} -A^* & 0 & C^* \\ \hline BB^* & A & -BD^* \\ \hline DB^* & C & I_p - DD^* \end{array} \right),$$

and if  $S$  is strictly contractive at infinity the inverse of  $I_p - DD^*$  is well defined. Then from (14), we have

$$(I_p - SS^*)^{-1} = \left( \begin{array}{cc|c} -A^* - C^*\Delta_l DB^* & -C^*\Delta_l C & C^*\Delta_l \\ \hline B\Delta_r B^* & A + BD^*\Delta_l C & -BD^*\Delta_l \\ \hline -\Delta_l DB^* & -\Delta_l C & \Delta_l \end{array} \right),$$

where  $\Delta_l = (I_p - DD^*)^{-1}$  and  $\Delta_r = (I_p - D^*D)^{-1}$ , whose dynamics matrix is none but  $H$ . Finally, it is easy to check from the definitions (21)-(23) that  $H$  is a Hamiltonian matrix, i.e. that the partition of  $H$  defined in (24) satisfies  $H_{12}^* = H_{12}$ ,  $H_{21}^* = H_{21}$ , and  $H_{22}^* = -H_{11}$ .  $\square$

**Remark 4.1** *The Hamiltonian character of  $H$  implies that it is similar to  $-H^*$ . In particular the eigenvalues of  $H$  are symmetric with respect to the imaginary axis, counting multiplicities. It must also be stressed that the realization of  $(I_p - SS^*)^{-1}$  given in the proof of Lemma 1 may not be minimal. Because the McMillan degree is invariant upon taking the inverse, the realization in question will in fact be minimal if, and only if, the McMillan degree of  $SS^*$  is the sum of the McMillan degrees of  $S$  and  $S^*$ . This will hold in particular when no zero of  $S$  is a pole of  $S^*$  [BGK], in other words if no zero of  $S$  is reflected from one of its poles. Hence the characteristic polynomial of  $H$  plays in the matrix-valued case the role of the polynomial  $\mu$  given by (16) in the scalar case (compare condition (ii) after (16)).*

The (not necessarily symmetric) inner extensions of  $S$  that preserve the McMillan degree are characterized by the following theorem borrowed from [GR]. Actually, theorem 4.1 in [GR] describes all the rational unitary (on the real line) extensions of a (non necessarily square) rational matrix function which is contractive on the real line and strictly contractive at infinity. The next theorem essentially rephrases this result in our right half plane setting dealing with square and stable matrix functions.

**Theorem 1** *If  $S$  given by (20) is a Schur function which is strictly contractive at infinity, then all  $(2p) \times (2p)$  inner extensions*

$$\mathcal{S} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S \end{pmatrix} \quad (25)$$

*of the same McMillan degree as  $S$  are given by*

$$\mathcal{S} = \begin{bmatrix} U_2 & 0 \\ 0 & I_p \end{bmatrix} \mathcal{S}_P \begin{bmatrix} U_1 & 0 \\ 0 & I_p \end{bmatrix} \quad (26)$$

*where  $U_1$  and  $U_2$  are arbitrary unitary matrices and where  $\mathcal{S}_P$  is given by*

$$\mathcal{S}_P = \left( \begin{array}{c|cc} A & B_1 & B \\ \hline C_1 & D_{11} & D_{12} \\ C & D_{21} & D \end{array} \right), \quad (27)$$

*with*

$$D_{21} = (I_p - DD^*)^{1/2}, \quad D_{12} = (I_p - D^*D)^{1/2}, \quad D_{11} = -D^*, \quad (28)$$

$$C_1 = -(I_p - D^*D)^{-1/2}(B^*P^{-1} + D^*C), \quad (29)$$

$$B_1 = -(PC^* + BD^*)(I_p - DD^*)^{-1/2}, \quad (30)$$

*and  $P$  is a Hermitian solution to the algebraic Riccati equation:*

$$\mathcal{R}(P) = P\hat{C}^*\hat{C}P + \hat{A}P + P\hat{A}^* + \hat{B}\hat{B}^* = 0. \quad (31)$$

*The map  $P \rightarrow \mathcal{S}_P$  is a one-to-one correspondence between the Hermitian solutions to (31) and the inner extensions of degree  $n$  of  $S$  whose value at infinity is  $\mathcal{D}$  defined in (27).*

**Remark 4.2** *Note that [GR, thm. 3.4.] guarantees, under the assumptions of Theorem 1, that all Hermitian solutions of (31) are invertible and positive definite since  $S(s)$  is stable.*



## 4.2 Relation to spectral factors.

We say that a stable  $p \times p$  matrix-valued function  $S_L$  (resp.  $S_R$ ) is a left (resp. right) spectral factor of  $I_p - SS^*$  (resp.  $I_p - S^*S$ ) if  $S_LS_L^* + SS^* = I_p$  (resp.  $S_R^*S_R + S^*S = I_p$ ); such a factor is called minimal if it is rational and if the block rational matrix  $(S_L \ S)$  (resp.  $(S_R^T \ S^T)^T$ ), whose McMillan degree is at least the degree of  $S$ , actually has the *same* McMillan degree as  $S$ . This is equivalent to require that  $(S_L \ S)$  (resp.  $(S_R^T \ S^T)^T$ ) has a minimal realization whose output (resp. input) and dynamics matrices are those of a minimal realization of  $S$ . In particular Theorem 1 implies that, for any inner extension of  $S$  having the same McMillan degree,  $S_{21}$  (resp.  $S_{12}$ ) is a minimal left (resp. right) spectral factor of  $I_p - SS^*$  (resp.  $I_p - S^*S$ ).

The corollary below is essentially a rephrasing of the theorem in terms of minimal spectral factors, compare [A1, F3, W]. Observe that substituting (28)-(30) in (27) yields

$$\begin{bmatrix} C_1 \\ C \end{bmatrix} = - \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D \end{bmatrix} \begin{bmatrix} B_1^* \\ B^* \end{bmatrix} P^{-1}$$

while a similar substitution in (31) yields

$$AP + PA^* + B_1B_1^* + BB^* = 0. \quad (32)$$

Note that (32) has a unique solution since  $A$  has no purely imaginary eigenvalue, and that this solution is necessarily Hermitian positive definite by the controllability of  $[A \ B]$ .

**Corollary 1** *Let  $S = \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$  be a minimal realization of a Schur function strictly contractive at infinity, and define  $D_{11}$ ,  $D_{21}$ , and  $D_{12}$  as in (28). To each minimal left spectral factor  $S_{21}$  of  $I_p - SS^*$  with value  $D_{21}$  at infinity, there is a unique inner extension of  $S$  of the same McMillan degree, whose lower left block is  $S_{21}$  and , with value at infinity:*

$$\mathcal{D} = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D \end{bmatrix}$$

If we put

$$[S_{21} \ S] = \left[ \begin{array}{c|cc} A & B_1 & B \\ \hline C & D_{21} & D \end{array} \right], \quad (33)$$

then this extension is none but  $\mathcal{S}_P$  given by (27) where  $P$  is the unique solution to (32). Alternatively, one also has

$$\mathcal{S}_P = \left[ \begin{array}{c|cc} A & B_1 & B \\ \hline - (D_{11}B_1^* + D_{12}B^*)P^{-1} & D_{11} & D_{12} \\ C & D_{21} & D \end{array} \right] \quad (34)$$

**Proof.** Suppose we have an inner extension  $\mathcal{S}$  of  $S$ , with the stated properties, given by (25). Then, from (32)-(33), the block  $S_{21}$  uniquely defines  $P$  thus also  $\mathcal{S}$  by Theorem 1. Conversely, let  $S_{21}$  be a minimal left spectral factor of  $I_p - SS^*$  with value  $D_{21}$  at infinity. Then we have a realization of the form (33) for  $[S_{21} \ S]$  and we may define  $P$  through (32). Using (13), we obtain for  $I_p - [S_{12} \ S][S_{12} \ S]^*$  the following realization

$$I_p - [S_{21} \ S][S_{21} \ S]^* = \left( \begin{array}{c|cc} -A^* & 0 & C^* \\ \hline B_1B_1^* + BB^* & A & -B_1D_{21}^* - BD^* \\ D_{21}B_1^* + DB^* & C & I_p - D_{21}D_{21}^* - DD^* \end{array} \right).$$

Performing the change of basis defined by  $\left[ \begin{array}{c|c} I_p & 0 \\ \hline -P & I_p \end{array} \right]$  using (28) and (32), we find another realization to be

$$I_p - [S_{21} \ S][S_{21} \ S]^* = \left( \begin{array}{cc|c} -A^* & 0 & C^* \\ 0 & A & -PC^* - B_1 D_{21}^* - BD^* \\ \hline CP + D_{21} B_1^* + DB^* & C & 0 \end{array} \right).$$

But the rational function under consideration is identically zero by definition of  $S_{21}$ , hence by the observability of  $[C \ A]$  we get in particular  $-PC^* - B_1 D_{21}^* - BD^* = 0$  which yields

$$C = -[D_{21} \ D][B_1 \ B]^* P^{-1}. \quad (35)$$

Now, put  $C_1 = -[D_{11} \ D_{12}][B_1 \ B]^* P^{-1}$  and let  $\mathcal{S}$  be defined as in (25). Starting from the realization of  $I_{2p} - \mathcal{S}\mathcal{S}^*$  provided by (13) and performing the change of basis defined by  $\left[ \begin{array}{c|c} I_p & 0 \\ \hline -P & I_p \end{array} \right]$ , a computation entirely similar to the previous one shows that this is the zero transfer function, that is,  $\mathcal{S}$  is inner. Moreover, (35) shows it is an extension of  $S$  whose lower left block is  $S_{21}$ , and clearly it has the same McMillan degree and value  $\mathcal{D}$  at infinity. Finally, it is straightforward to check that this extension satisfies (34).  $\square$

An inner extension  $\mathcal{S}_P$  of  $S$  with the same McMillan degree and value  $\mathcal{D}$  at infinity is thus completely determined by the choice of a minimal left spectral factor of  $I_p - \mathcal{S}\mathcal{S}^*$ . Of course a dual result holds true on the right, namely the extension is also uniquely determined by a right minimal spectral factor  $S_{12}$  of  $I_p - S^*S$ . In what follows, we only deal with inner extensions having value  $\mathcal{D}$  at infinity, which the normalization induced by (26) on letting  $U_1 = U_2 = I_p$  there.

Let  $P$  be a solution of the Riccati equation (31). Then the matrix  $H$  defined in (24) satisfies the similarity relation

$$\left[ \begin{array}{cc} I_p & 0 \\ -P & I_p \end{array} \right] H \left[ \begin{array}{cc} I_p & 0 \\ P & I_p \end{array} \right] = \left[ \begin{array}{cc} -(\hat{A} + P\hat{C}^*\hat{C})^* & -\hat{C}^*\hat{C} \\ 0 & \hat{A} + P\hat{C}^*\hat{C} \end{array} \right]. \quad (36)$$

Thus

$$\chi_H(s) = \chi_Z(s)\chi_{-Z^*}(s), \quad (37)$$

where  $\chi_M$  denotes the characteristic polynomials of  $M$  and where we have set

$$Z = \hat{A} + P\hat{C}^*\hat{C}. \quad (38)$$

Moreover, since  $S$  is strictly contractive at infinity,  $S_{21}$  is invertible and, in view of (14),  $S_{21}^{-1}$  has the dynamics matrix

$$A - B_1 D_{21}^{-1} C = A + BD^*(I_p - DD^*)^{-1}C + PC^*(I_p - DD^*)^{-1}C = \hat{A} + P\hat{C}^*\hat{C} = Z. \quad (39)$$

Likewise,  $S_{12}^{-1}$  has the dynamics matrix

$$A - BD_{12}^{-1}C_1 = -P(\hat{A} + P\hat{C}^*\hat{C})^* P^{-1} = -PZ^* P^{-1}. \quad (40)$$

This way the extension process is seen to divide out the eigenvalues of  $H$  between the inverses of the left and right spectral factors of  $I_p - \mathcal{S}\mathcal{S}^*$  and  $I_p - S^*S$  respectively.

It is a classical fact that there exists a natural partial ordering on the set of Hermitian solutions to (31), namely  $P_1 \leq P_2$  if and only if the difference  $P_2 - P_1$  is positive semidefinite. It is well-known (see [LR1, sect.2.5.] or [LR2]) that there exists a maximal solution  $\hat{P}$  and a minimal solution  $\check{P}$  of (31):  $\hat{P}$  is the unique solution for which  $\sigma(\hat{A} + \hat{P}\hat{C}^*\hat{C}) \subset \overline{\Pi}^+$ , while  $\check{P}$  is the unique solution for which  $\sigma(\hat{A} + \check{P}\hat{C}^*\hat{C}) \subset \overline{\Pi}^-$ , where  $\sigma(M)$  denotes the spectrum of  $M$ . The left spectral factor  $\check{S}_{21}$  associated with  $\check{P}$  is called the *outer spectral factor*. Its inverse is analytic in  $\Pi^+$ .

**Proposition 1** *Let  $S$  be a Schur function which is strictly contractive at infinity. Let  $P$  and  $\tilde{P}$  be two distinct solutions of (31) and*

$$\mathcal{S}_P = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S \end{bmatrix}, \quad \mathcal{S}_{\tilde{P}} = \begin{bmatrix} \tilde{S}_{11} & \tilde{S}_{12} \\ \tilde{S}_{21} & S \end{bmatrix},$$

*the inner extensions of  $S$  associated to them via Theorem 1.*

*Then, the matrix  $Q = S_{21}^{-1}\tilde{S}_{21}$  is well-defined and unitary on the imaginary axis. Its McMillan degree coincides with the rank of  $\tilde{P} - P$  and  $Q$  is inner if and only if  $\tilde{P} \geq P$ .*

**Proof.** Assuming that  $S$  is strictly contractive at infinity,  $S_{21}$  is invertible and we may define  $Q = S_{21}^{-1}\tilde{S}_{21}$ . We have that

$$QQ^* = S_{21}^{-1}\tilde{S}_{21}\tilde{S}_{21}^*S_{21}^{-*} = S_{21}^{-1}(I_p - SS^*)S_{21}^{-*} = S_{21}^{-1}(S_{21}S_{21}^*)S_{21}^{-*} = I_p, \quad (41)$$

so that  $Q$  is unitary. A realization of  $Q$  can be computed from the realizations of  $S_{21}$  and  $\tilde{S}_{21}$  of Theorem 1, using (13) and (14):

$$Q = \left( \begin{array}{c|c} A - B_1D_{21}^{-1}C & B_1D_{21}^{-1} \\ \hline -D_{21}^{-1}C & D_{21}^{-1} \end{array} \right) \left( \begin{array}{c|c} A & \tilde{B}_1 \\ \hline C & D_{21} \end{array} \right) = \left( \begin{array}{cc|c} A - B_1D_{21}^{-1}C & B_1D_{21}^{-1}C & B_1 \\ 0 & A & \tilde{B}_1 \\ \hline -D_{21}^{-1}C & D_{21}^{-1}C & I_p \end{array} \right).$$

Applying a change of variables using  $T = \begin{bmatrix} I_n & -I_n \\ 0 & I_n \end{bmatrix}$  we get

$$Q = \left( \begin{array}{cc|c} A - B_1D_{21}^{-1}C & 0 & B_1 - \tilde{B}_1 \\ 0 & A & \tilde{B}_1 \\ \hline -D_{21}^{-1}C & 0 & I_p \end{array} \right) = \left( \begin{array}{c|c} A - B_1D_{21}^{-1}C & B_1 - \tilde{B}_1 \\ \hline -D_{21}^{-1}C & I_p \end{array} \right). \quad (42)$$

From (30) we draw  $B_1 - \tilde{B}_1 = (\tilde{P} - P)C^*D_{21}^{-1}$  and

$$Q = \left( \begin{array}{c|c} Z & (\tilde{P} - P)C^*D_{21}^{-1} \\ \hline -D_{21}^{-1}C & I_p \end{array} \right) \quad (43)$$

where  $Z$  is given by (38).

Set  $\Gamma = \tilde{P} - P$ . Since  $\Gamma$  is Hermitian, the singular value decomposition can be written

$$\Gamma = V \begin{bmatrix} 0 & 0 \\ 0 & \Gamma_0 \end{bmatrix} V^*, \quad (44)$$

where  $V$  is unitary and  $\Gamma_0$  is real and diagonal. Let  $d \times d$  be the size of  $\Gamma_0$ , and partition  $V$  as  $V = [V_1 \ V_2]$  where  $V_1$  is of size  $n \times (n - d)$  and  $V_2$  of size  $n \times d$ . Note that the columns of  $V_1$  span the kernel of  $\Gamma$ . In another connection, it holds that

$$\begin{aligned}\mathcal{R}(\tilde{P}) - \mathcal{R}(P) &= \hat{A}\Gamma + \Gamma\hat{A}^* + \tilde{P}\hat{C}^*\hat{C}\tilde{P} - P\hat{C}^*\hat{C}P \\ &= Z\Gamma - P\hat{C}^*\hat{C}\Gamma + \Gamma Z^* - \Gamma\hat{C}^*\hat{C}P + \Gamma\hat{C}^*\hat{C}\tilde{P} + P\hat{C}^*\hat{C}\Gamma \\ &= Z\Gamma + \Gamma Z^* + \Gamma\hat{C}^*\hat{C}\Gamma,\end{aligned}$$

which is zero since both  $P$  and  $\tilde{P}$  are solutions to the Riccati equation (31). Thus  $\Gamma$  is a solution to the Riccati equation

$$Z\Gamma + \Gamma Z^* + \Gamma\hat{C}^*\hat{C}\Gamma = 0. \quad (45)$$

Partitioning  $V^*ZV$  into

$$V^*ZV = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}, \quad (46)$$

we can rewrite (45) as

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \Gamma_0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \Gamma_0 \end{bmatrix} \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}^* + \begin{bmatrix} 0 & 0 \\ 0 & \Gamma_0 \end{bmatrix} V^*\hat{C}^*\hat{C}V \begin{bmatrix} 0 & 0 \\ 0 & \Gamma_0 \end{bmatrix} = 0,$$

that is

$$\begin{bmatrix} 0 & Z_{12}\Gamma_0 \\ \Gamma_0 Z_{12}^* & Z_{22}\Gamma_0 + \Gamma_0 Z_{22}^* + \Gamma_0 V_2^*\hat{C}^*\hat{C}V_2\Gamma_0 \end{bmatrix} = 0.$$

Therefore

$$Z_{12} = 0 \quad (47)$$

and

$$Z_{22}\Gamma_0 + \Gamma_0 Z_{22}^* + \Gamma_0 V_2^*\hat{C}^*\hat{C}V_2\Gamma_0 = 0. \quad (48)$$

Using  $V$  as change of coordinates in the state space, we obtain a new realization for  $Q$ :

$$Q = \left( \begin{array}{c|c} V^*ZV & D_{21}^{-1}V^*\Gamma C^* \\ \hline CV D_{21}^{-1} & I_p \end{array} \right) = \left( \begin{array}{cc|c} Z_{11} & 0 & 0 \\ Z_{21} & Z_{22} & \Gamma_0 V_2^* C^* D_{21}^{-1} \\ \hline D_{21}^{-1}CV_1 & D_{21}^{-1}CV_2 & I_p \end{array} \right) \quad (49)$$

where we used (47). This readily reduces to

$$Q = \left( \begin{array}{c|c} Z_{22} & \Gamma_0 V_2^* C^* D_{21}^{-1} \\ \hline D_{21}^{-1}CV_2 & I_p \end{array} \right), \quad (50)$$

and by (48) we also have (recall  $\Gamma_0$  is invertible) that  $\Gamma_0^{-1}$  solves the Lyapunov equation

$$Z_{22}^*\Gamma_0^{-1} + \Gamma_0^{-1}Z_{22} + V_2^*\hat{C}^*\hat{C}V_2 = 0. \quad (51)$$

From (50), it is clear that the degree of  $Q$  is at most the size of  $\Gamma_0$  which is the rank of  $\Gamma = \tilde{P} - P$ . We claim that the realization (50) is minimal. Indeed, it is observable because we started from the observable realization (42) and we just restricted ourselves in step (49)-(50) to some invariant subspace of the dynamics matrix in the state space. To check reachability, we use Hautus's test that no nonzero left eigenvector  $x^T$  of  $Z_{22}$  associated, say, to some eigenvalue  $\lambda$ , can lie in the left

kernel of  $\Gamma_0 V_2^* C^* D_{21}^{-1}$ ; for then (48) and (23) together imply that  $x^T \Gamma_0$ , which is nonzero as  $\Gamma_0$  has full rank, is a left eigenvector of  $Z_{22}^*$  associated to  $-\lambda$ . Thus  $\Gamma_0^* \bar{x}$  would be an eigenvector of  $Z_{22}$  and by construction it lies in the kernel of  $D_{21}^{-1} C V_2$ , contradicting the observability of (50). *This proves the claim*, to the effect that the degree of  $Q$  is in fact equal to the rank of  $\Gamma = \tilde{P} - P$ . Now, from classical properties of solutions to Lyapunov equations [BGR, th.6.5.2], the number of poles of  $Q$  in  $\Pi^+$  is equal to the number of negative eigenvalues of  $\Gamma_0^{-1}$  solving (51).  $\square$

**Remark 4.3** *A similar result holds true for the right inner factors: the matrix  $R = \tilde{S}_{12} S_{12}^{-1}$  is unitary on the imaginary axis and inner if and only if  $\tilde{P} \leq P$ .*

The outer spectral factor play an important role in what follows, due to the fact that *any* (not necessarily minimal) left spectral factor, say,  $\sigma$  of  $I_p - S S^*$  can be factored as  $\sigma = \check{S}_{21} Q$  where  $Q$  is inner. Indeed, the strict contractivity of  $S$  at infinity entails that  $\sigma$  is invertible as a rational matrix, and then computation (41) with  $S_{21}, \tilde{S}_{21}$  replaced by  $\check{S}_{21}, \sigma$  shows that the rational matrix  $Q = \check{S}_{21}^{-1} \sigma$  is unitary on the imaginary axis. In particular it cannot have a pole there, and since it is analytic in  $\Pi^+$  as  $\sigma$  is stable and  $\check{S}_{21}$  outer, we conclude that  $Q$  is stable thus inner, as desired.

### 4.3 Inner extensions of higher degree

We shall be interested in inner rational extensions where we allow for an increase in the McMillan degree, and we will base our analysis on the next proposition. For the proof, we need a notion of coprimeness: two inner-functions  $L_1, L_2$  are *right coprime* if one cannot write  $L_1 = G_1 J$  and  $L_2 = G_2 J$  with  $G_1, G_2, J$  some inner functions and  $J$  non-constant. It is well known [F1, BO1] that this is equivalent to require the existence of two stable rational matrix functions  $X_1$  and  $X_2$  such that the following Bezout equation holds:

$$X_1(s) L_1(s) + X_2(s) L_2(s) = I_p. \quad (52)$$

Left coprimeness is defined in a symmetric way.

**Proposition 2** *All rational inner extensions of a Schur function  $S$ , contractive at  $\infty$ , can be written on the form*

$$\begin{bmatrix} L & 0 \\ 0 & I_p \end{bmatrix} \mathcal{S}_P \begin{bmatrix} R & 0 \\ 0 & I_p \end{bmatrix} \quad (53)$$

where  $L, R$  and  $\mathcal{S}_P$  are inner, and  $\mathcal{S}_P$  is an extension of  $S$  at the same McMillan degree, obtained from a solution  $P$  of the Riccati equation (31).

**Proof.** Let

$$\check{S} = \begin{bmatrix} \check{S}_{11} & \check{S}_{12} \\ \check{S}_{21} & S \end{bmatrix}$$

be the inner extension of  $S$  of degree  $n$  associated with the minimal solution  $\check{P}$  of (31), so that  $\check{S}_{21}$  is the outer left spectral factor of  $I_p - S S^*$ . For an arbitrary rational inner extension

$$\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & S \end{bmatrix}$$

of  $S$ , we mechanically obtain since  $\Sigma\Sigma^* = \check{S}\check{S}^* = I_{2p}$  that

$$\check{S}_{21}\check{S}_{21}^* = I_p - SS^* = \sigma_{21}\sigma_{21}^* \quad \text{and} \quad \check{S}_{12}^*\check{S}_{12} = I_p - S^*S = \sigma_{12}^*\sigma_{12}.$$

Therefore, computing as in (41), we can write  $\sigma_{21} = \check{S}_{21}R$  and  $\sigma_{12} = L\check{S}_{12}$  where  $R = \check{S}_{21}^{-1}\sigma_{21}$  and  $L = \sigma_{12}\check{S}_{12}^{-1}$  are rational matrices that are unitary on the imaginary axis. By the discussion after Remark 4.3,  $R$  is inner. Next, using again that  $\Sigma\Sigma^* = I_{2p}$ , we get  $\sigma_{11} = -\sigma_{12}S^*\sigma_{21}^{-*} = -L\check{S}_{12}S^*\check{S}_{21}^{-*}R = L\check{S}_{11}R$ , and therefore

$$\Sigma = \begin{bmatrix} L & 0 \\ 0 & I_p \end{bmatrix} \begin{bmatrix} \check{S}_{11} & \check{S}_{12} \\ \check{S}_{21} & S \end{bmatrix} \begin{bmatrix} R & 0 \\ 0 & I_p \end{bmatrix}.$$

If  $L$  is inner, we have finished because (53) holds with  $P = \check{P}$ . Otherwise, being a unitary rational function,  $L$  can be factored as  $L = L_1L_2^*$  where  $L_1$  and  $L_2$  are right coprime inner functions, so that (52) holds for some stable transfer functions  $X_1, X_2$ . In fact, the existence of such a factorization follows from the so-called Douglas-Shapiro-Shields factorization [DSS] as carried over to matrix-valued strictly non cyclic functions in [F1], see *e.g.* [BO1] for a detailed discussion of the rational case in discrete time that translates immediately to continuous time by linear fractional transformation.

From (52) we deduce that  $L_2^*(s) = X_1(s)L(s) + X_2(s)$ , therefore

$$\begin{bmatrix} L_2^* & 0 \\ 0 & I_p \end{bmatrix} \begin{bmatrix} \check{S}_{11} & \check{S}_{12} \\ \check{S}_{21} & S \end{bmatrix} \begin{bmatrix} R & 0 \\ 0 & I_p \end{bmatrix} = \begin{bmatrix} X_1 & 0 \\ 0 & I_p \end{bmatrix} \Sigma + \begin{bmatrix} X_2 & 0 \\ 0 & I_p \end{bmatrix} \check{S} \begin{bmatrix} R & 0 \\ 0 & I_p \end{bmatrix} \quad (54)$$

is stable. In particular, if we put  $S_{12} := L_2^*\check{S}_{12}$ , we see that

$$\begin{bmatrix} S_{12} \\ S \end{bmatrix} = \begin{bmatrix} L_2^* & 0 \\ 0 & I_p \end{bmatrix} \begin{bmatrix} \check{S}_{12} \\ S \end{bmatrix} \quad (55)$$

is stable. Now, multiplication by a rational function analytic in  $\overline{\Pi}^-$  (including at infinity) followed by the projection onto stable rational functions (obtained by partial fraction extension) *cannot increase the McMillan degree*; this follows at once from Fuhrmann's realization theory [F1, BO1]. Therefore, we deduce from (55) that the degree of  $[S_{12}^T \ S^T]^T$  is at most the degree of  $[\check{S}_{12}^T \ S^T]^T$ . But the latter is equal to  $n$  for Corollary 1 as applied to  $\check{S}^T$  implies that  $\check{S}_{12}$  is a minimal right spectral factor of  $I_p - S^*S$ . Hence the degree of  $[S_{12}^T \ S^T]^T$  is  $n$  and  $S_{12}$  is again a minimal right spectral factor of  $I_p - S^*S$ . Thus by (the transposed version of) Corollary 1, there exists an inner extension of  $S$  of McMillan degree  $n$  associated with  $S_{12}$ :

$$\mathcal{S}_P = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S \end{bmatrix}.$$

As seen after Remark 4.3, one has  $S_{21} = \check{S}_{21}R_2$  for some inner matrix  $R_2$ . Moreover, by the inner character of  $\mathcal{S}_P$  and  $\mathcal{S}_{\check{P}}$  and in view of (55), we get

$$S_{11} = -S_{12}S^*S_{21}^{-*} = -L_2^*\check{S}_{12}S^*\check{S}_{21}^{-*}R_2^{-*} = L_2^*\check{S}_{11}R_2^{-*}.$$

Altogether, this implies

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S \end{bmatrix} \begin{bmatrix} R_2^* & 0 \\ 0 & I_p \end{bmatrix} = \begin{bmatrix} L_2^* & 0 \\ 0 & I_p \end{bmatrix} \begin{bmatrix} \check{S}_{11} & \check{S}_{12} \\ \check{S}_{21} & S \end{bmatrix}. \quad (56)$$

Next, *we contend* that the inner functions  $\mathcal{S}_P$  and  $\text{diag}[R_2, I_p]$  are right coprime. Indeed, if  $R_c$  is a right common inner factor, we get in particular  $R_c = \text{diag}[R_2, I_p]J^*$  for some inner  $J$ , so the last  $p$  rows of both  $J$  and  $J^*$  are analytic in  $\bar{\Pi}^+$ , *i.e.* they are constant. This entails

$$R_c = U \begin{bmatrix} R_3 & 0 \\ 0 & I_p \end{bmatrix} \quad (57)$$

for some constant unitary matrix  $U$ . But then  $\mathcal{S}_P R_c^* U$  is inner since  $R_c$  divides  $\mathcal{S}_P$ , and it is an extension of  $S$  by (57). Taking determinants, we see that this extension has McMillan degree equal to  $\deg \mathcal{S}_P - \deg R_3$ . As this degree is at least equal to that of  $S$  which is also that of  $\mathcal{S}_P$ , we conclude that  $R_3$  thus also  $R_c$  are constant, *which proves our contention*.

By the coprimeness above, there exist stable rational matrices  $X$  and  $Y$  such that

$$X \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S \end{bmatrix} + Y \begin{bmatrix} R_2 & 0 \\ 0 & I_p \end{bmatrix} = I_{2p}. \quad (58)$$

From (54)-(56), we see that the product  $\mathcal{S}_P \text{diag}[R_2^* R, I_p]$  is stable. Therefore, right multiplying (58) by  $\text{diag}[R_2^* R, I_p]$ , we deduce that the latter is also stable hence  $R_1 := R_2^* R$  is inner. Finally,

$$\Sigma = \begin{bmatrix} L_1 & 0 \\ 0 & I_p \end{bmatrix} \mathcal{S}_P \begin{bmatrix} R_1 & 0 \\ 0 & I_p \end{bmatrix}$$

is indeed of the form (53), as wanted.  $\square$

## 5 Symmetric unitary extensions

We assume from now on that the Schur function  $S$  is symmetric *i.e.*  $S = S^T$  and we consider a symmetric realization,

$$S = \left( \frac{A \mid B}{C \mid D} \right), \quad A = A^T, \quad B = C^T, \quad D = D^T \quad (59)$$

Such a realization always exists thanks to Theorem 5 in [FH].

It follows that the matrices  $\hat{A}, \hat{B}\hat{B}^*, \hat{C}^*\hat{C}$  defined by (21), (22) and (23) satisfy

$$\begin{cases} \hat{A} &= \hat{A}^T \\ \hat{B}\hat{B}^* &= (\hat{C}^*\hat{C})^T. \end{cases} \quad (60)$$

If the extension  $\mathcal{S}_P$  associated to some solution  $P$  of (31) *via* Theorem 1 is symmetric, then the matrix  $Z$  defined in (38) must be similar to  $-Z^*$  since they are, by the computations (39) and (40), dynamics matrices of  $S_{21}^{-1}$  and  $S_{12}^{-1}$  respectively. Therefore, in view of (37), the characteristic polynomial  $\chi_H$  must be of the form  $\pi(s)^2$ . As this may not be the case, a symmetric inner extension of  $S$  preserving the McMillan degree may well fail to exist. However, as we shall see (*cf.* also [AV]), symmetric inner extensions of higher degree do exist. We will first give a simple method to construct (possibly unstable) symmetric unitary extensions of  $S$  using the inner extensions  $\mathcal{S}_P$  provided to us by Theorem 1.

As the Riccati equation (31) does admit a Hermitian solution and since the pair  $(\hat{A}, \hat{C})$  in (21)-(23) is observable, as follows immediately from Hautus's test on using the observability of  $(A, C)$ ,

the partial multiplicities (*i.e.* the sizes of the Jordan blocks) of the pure imaginary eigenvalues of  $H$  (if any) are all even, see [LR1, th.2.6] or [LR2, th. 7.3.1]. Let  $2n_0$  be the dimension of the spectral subspace of  $H$  corresponding to all its pure imaginary eigenvalues. Using (36) and grouping together the roots of even multiplicity, we can write the characteristic polynomial of  $H$  in the form

$$\chi_H(s) = \pi(s)^2 \chi_\kappa^+(s) \chi_\kappa^-(s), \quad (61)$$

where  $\pi(s)$ ,  $\chi_\kappa^+(s)$ , and  $\chi_\kappa^-(s)$  are polynomials in  $s$ , and where  $\chi_\kappa^+$  has  $\kappa$  simple roots in  $\Pi^+$  while  $\chi_\kappa^- = (\chi_\kappa^+)^*$ . Then  $\kappa$  is the number of distinct eigenvalues of  $H$  in  $\Pi^+$  with odd multiplicity, and  $\pi(s)$  has degree greater than or equal to  $n_0$  by what precedes.

**Proposition 3** *Assume  $S$  is a symmetric Schur function strictly contractive at infinity, and let (59) be a symmetric realization. For  $P$  a Hermitian solution of (31), let further*

$$\mathcal{S}_P = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S \end{bmatrix},$$

*be the inner extension of the same degree associated by Theorem 1.*

*Then, the matrix  $Q = S_{21}^{-1} S_{12}^T$  is well-defined, unitary on the imaginary axis, and*

$$\Sigma_P = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & I_p \end{bmatrix} = \begin{bmatrix} S_{11} S_{21}^{-1} S_{12}^T & S_{12} \\ S_{12}^T & S \end{bmatrix},$$

*is a symmetric extension of  $S$  which is unitary on the imaginary axis. The McMillan degree of  $Q$  coincides with the rank of  $P^{-T} - P$ , and  $Q$  is inner if and only if  $P^{-T} - P$  is positive semi-definite. In this case,  $\Sigma_P$  is inner and its McMillan degree is  $\deg S + \deg Q$ .*

*Moreover,  $\deg Q$  is in any case greater than or equal to  $\kappa$ , the number of distinct eigenvalues of  $H$  in  $\Pi^+$  with odd multiplicity.*

**Proof.** From (60) we see that  $\mathcal{R}(P) = 0$  if and only if  $\mathcal{R}(P^{-T}) = 0$ , and that  $\mathcal{S}_{P^{-T}} = \mathcal{S}_P^T$ . Therefore, we may appeal to Proposition 1 with  $\tilde{P} = P^{-T}$  and  $\tilde{S}_{21} = S_{12}^T$ . Thus,  $Q = S_{21}^{-1} S_{12}^T$  is unitary on the imaginary axis, its degree  $d$  is equal to the rank of  $P^{-T} - P$  and  $Q$  is inner if and only if  $P^{-T} \geq P$ . The computations in the proof of Proposition 1 apply, so if (44) is the singular value decomposition of  $\Gamma = P^{-T} - P$  and  $Z$  is defined by (38), we get from (46) and (47) that

$$V^* Z V = \begin{bmatrix} V_1^* Z V_1 & V_1^* Z V_2 \\ V_2^* Z V_1 & V_2^* Z V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & 0 \\ Z_{21} & Z_{22} \end{bmatrix}. \quad (62)$$

On the other hand, if we set  $\tilde{Z} := -P Z^* P^{-1}$ , it follows from (31) and (38) that

$$\tilde{Z} = -P Z^* P^{-1} = -P(\hat{A}^* + \hat{C}^* \hat{C} P)^* P^{-1} = (\hat{A} P + \hat{B} \hat{B}^*)^* P^{-1} = \hat{A} + \hat{B} \hat{B}^* P^{-1}. \quad (63)$$

But since the columns of  $V_1$  span the kernel of  $\Gamma$ , we get  $V_1^*(P - P^{-T}) = 0$  that transposes into  $P^T \bar{V}_1 = P^{-1} \bar{V}_1$ , hence in view of (63) and (60)

$$\tilde{Z} \bar{V}_1 = (\hat{A} + \hat{B} \hat{B}^* P^T) \bar{V}_1 = Z^T \bar{V}_1.$$

As  $V$  is unitary, we have thus arrived at a similarity relation of the form:

$$V^T \tilde{Z} \bar{V} = \begin{bmatrix} V_1^T \tilde{Z} \bar{V}_1 & V_1^T \tilde{Z} \bar{V}_2 \\ V_2^T \tilde{Z} \bar{V}_1 & V_2^T \tilde{Z} \bar{V}_2 \end{bmatrix} = \begin{bmatrix} Z_{11}^T & \tilde{Z}_{12} \\ 0 & \tilde{Z}_{22} \end{bmatrix}. \quad (64)$$



Now, since  $\tilde{Z}$  is similar to  $-Z^*$ , we see from (62), (64), and (37) that  $\chi_{Z_{11}}^2(s)$  must divide  $\chi_H(s)$ . Consequently the number of rows (or columns) of  $Z_{11}$ , which is the dimension of the kernel of  $P^{-T} - P$ , cannot exceed the degree of  $\pi(s)$  in (61), namely  $n - \kappa$ . Therefore  $\text{rank}(P^{-T} - P) \geq \kappa$ , as announced.  $\square$

**Remark 5.1** Note that  $\deg Q = 0$ , i.e.  $Q$  is constant, if and only if  $P^{-T} = P$ .

## 6 Minimal symmetric inner extensions

**Lemma 2** Let  $\check{P}$  and  $\hat{P}$  be the minimal and the maximal solution to the Riccati equation (31) associated with a symmetric Schur function strictly contractive at infinity. Then

$$\check{P}^{-T} = \hat{P}.$$

**Proof.** Since  $\check{P}$  is the minimal solution, for each solution  $P$  we have that  $P \geq \check{P}$ . By symmetry,  $P$  is a solution if and only if  $P^{-T}$  is a solution and moreover  $\check{P}^{-T} \geq P^{-T}$ , so that  $\check{P}^{-T}$  must be the maximal solution.  $\square$

A symmetric inner extension of the symmetric Schur function  $S$  may now be obtained as follows.

**Proposition 4** Let

$$S_{\check{P}} = \begin{bmatrix} \check{S}_{11} & \check{S}_{12} \\ \check{S}_{21} & S \end{bmatrix}$$

be the extension associated with the minimal solution  $\check{P}$  to the Riccati equation (31). The symmetric extension  $\Sigma_{\check{P}}$  given by Proposition 3:

$$\Sigma_{\check{P}} = \begin{bmatrix} \check{S}_{11} & \check{S}_{12} \\ \check{S}_{21} & S \end{bmatrix} \begin{bmatrix} \check{Q} & 0 \\ 0 & I_p \end{bmatrix}, \quad \text{with } \check{Q} = \check{S}_{21}^{-1} \check{S}_{12}^T,$$

is inner and has degree  $2n - n_0$ .

**Proof.** Since  $\check{P}$  is the minimal solution of the Riccati equation, we have that  $\check{P}^{-T} - \check{P} \geq 0$ , hence by Proposition 3 the extension  $\Sigma_{\check{P}}$  is inner and has degree  $n + d$  where  $d$  is the rank of  $\check{P}^{-T} - \check{P}$ . By Lemma 2  $\check{P}^{-T} = \hat{P}$ , and if we let  $\ker(\hat{P} - \check{P}) = \mathcal{N}$  it is a classical fact (see e.g. [LR1, th.2.12] or [LR2]) that  $\mathcal{N}$  is, for any solution  $P$  to (31), the spectral subspace of  $Z = \hat{A} + P\hat{C}^*\hat{C}$  corresponding to all of its pure imaginary eigenvalues, and that it has dimension  $n_0$ . Therefore  $\deg \Sigma_{\check{P}} = 2n - n_0$ , as desired.  $\square$

Finally, we shall construct from  $\Sigma_{\check{P}}$  a symmetric inner extension of  $S$  of minimal degree. For this, we first establish a factorization property which strengthens the Potapov factorization (7) in the symmetric case.

For  $\text{Re } \xi > 0$ , recall from (5) and (6) the definitions of  $b_\xi$  and  $B_\xi$ . Let  $U$  be a unitary matrix whose first column is a unit vector  $u \in \mathbb{C}^p$ , then

$$B_{\xi,u} = UB_\xi(s)U^* = I_p + (b_\xi(s) - 1)uu^* \quad (65)$$

is an elementary Blaschke factor with the following properties (compare [D4, chap.1]):

$$B_{\xi,u}^*(s) = I_p + (b_\xi(s)^{-1} - 1)uu^* \quad (66)$$

$$\det B_{\xi,u}(s) = b_\xi(s). \quad (67)$$

Note that any elementary Blaschke factor can be written in the form  $B_{\xi,u}V$  for some unit vector  $u$  and some unitary matrix  $V$ .

**Lemma 3** *For  $F$  a  $p \times p$  symmetric complex matrix, there exists a unitary matrix  $U$  such that*

$$F = U\Lambda U^T \quad (68)$$

*with  $\Lambda$  a non-negative diagonal matrix. If  $F$  has rank  $k$ , the  $p-k$  first columns of  $U$  may be construed to be any orthonormal basis of the kernel of  $F$ .*

**Proof.** The existence of (68) is proved in [HJ, cor.4.4.4] under the name of Takagi's factorization. Permuting the columns of  $U$ , we may arrange the diagonal entries of  $\Lambda$  in non-decreasing order and then the first  $p-k$  columns form an orthogonal basis of  $\text{Ker}F$ . Right multiplying  $U$  by a block diagonal unitary matrix of the form  $\text{diag}[V, I_k]$  we may clearly trade that basis for any other.  $\square$

**Lemma 4** *Let  $T(s)$  be a symmetric inner function and  $\xi \in \Pi^+$  a zero of  $T(s)$ . Suppose that there exists a unit vector  $u$  such that the interpolation conditions*

$$T(\xi)u = 0 \quad (69)$$

$$u^T T'(\xi)u = 0 \quad (70)$$

*are satisfied. Defining  $B_{\xi,u}(s)$  as in (65), then,*

$$R(s) = B_{\xi,u}(s)^{-T} T(s) B_{\xi,u}(s)^{-1},$$

*is analytic at  $\xi$  and thus a symmetric inner function of degree  $N-2$ .*

**Proof.** We give a proof of this result which follows Potapov's approach to the multiplicative structure of  $J$ -inner functions [P]. For simplicity, we use Landau's notation  $O(s)^k$  for the class of (scalar or matrix-valued) functions  $f(s)$  such that  $\|f(s)\|/|s|^k$  is bounded for  $|s|$  sufficiently small. We also put  $\|f(s)\|$  for the operator norm of  $f(s)$ . Write the Taylor expansion of  $T(s)$  about  $\xi$  as

$$T(s) = T(\xi) + (s - \xi)T'(\xi) + O(s - \xi)^2,$$

and apply Lemma 3 to obtain Takagi's factorization of  $T(\xi)$  in the form

$$U^T T(\xi) U = \text{diag}(0, \dots, 0, \rho_1, \dots, \rho_r), \quad (71)$$

for some unitary matrix  $U$  whose first column is  $u$ . Next, define a matrix-valued function  $T_1$  by

$$\begin{aligned} T_1(s) &:= T(s)B_{\xi,u}(s)^{-1} \\ &= T(\xi)B_{\xi,u}(s)^{-1} + (s - \xi)T'(\xi)B_{\xi,u}(s)^{-1} + O(s - \xi) \\ &= (U^*)^T \text{diag}(0, \dots, 0, \rho_1, \dots, \rho_r) B_{\xi,u}(s)^{-1} U^* + (s - \xi)T'(\xi)B_{\xi,u}(s)^{-1} + O(s - \xi) \\ &= T(\xi) + (s - \xi)T'(\xi)B_{\xi,u}(s)^{-1} + O(s - \xi) \end{aligned}$$

where we have used (65). Clearly,  $T_1(s)$  is analytic about  $\xi$  and, since  $B_{\xi,u}(s)$  has degree 1,  $T_1(s)$  is an inner function of McMillan degree  $N-1$ . Now, using (66) and the symmetry of  $T(\xi)$  which implies  $u^T T(\xi) = 0$ , we get the following interpolation condition for  $T_1(s)$ :

$$u^T T_1(\xi) = u^T T(\xi) + 2\text{Re } \xi u^T T'(\xi) u u^* = 0.$$

Applying what precedes to  $T_1(\xi)^T$ , we see that  $R(s) := T_1(\xi)^T B_{\xi,u}(s)^{-1}$  is analytic at  $\xi$  and thus an inner function of degree  $N - 2$ . Finally, we can write

$$R(s) = B_{\xi,u}(s)^{-T} T(s) B_{\xi,u}(s)^{-1},$$

which achieves the proof.  $\square$

**Proposition 5** *Let  $T(s)$  be a symmetric inner function of degree  $N$  and suppose that  $T(s)$  has a zero  $\xi \in \Pi^+$  of multiplicity greater than 1. Then there exists an elementary Blaschke factor of the form  $B(s) = B_{\xi,u}$ , where  $u \in \mathbb{C}^p$  is a unit vector in the kernel of  $T(\xi)$ , such that*

$$R(s) = B(s)^{-T} T(s) B(s)^{-1}$$

*is analytic at  $\xi$ . Thus  $R(s)$  is inner, symmetric, and it has McMillan degree  $N - 2$ . Moreover, if  $\mathcal{V}$  is a 2-dimensional subspace of the kernel of  $T(\xi)$ , we may impose in addition that  $u \in \mathcal{V}$ .*

**Proof.** Write the local Smith form of  $T$  at  $\xi$ , namely (11) where  $S$  is replaced by  $T$  and  $k$  is equal to  $p$ . Assume first that the kernel of  $T(\xi)$  has dimension 1 over  $\mathbb{C}$ . Since  $T$  is analytic at  $\xi$  and  $\xi$  is a zero of multiplicity at least 2, the partial multiplicities must satisfy  $\nu_j = 0$  for  $1 \leq j < p$  and  $\nu_p \geq 2$ . Therefore, if  $e_p$  is the last element of the canonical basis of  $\mathbb{C}^p$ , the vector  $u_0 = F^{-1}(\xi)e_p$  spans the kernel of  $T(\xi)$  and the  $\mathbb{C}^p$ -valued function  $\phi(s) = F^{-1}(s)e_p$  is such that  $T(s)\phi(s)$  has a zero of order at least 2 at  $\xi$ :

$$T(s)\phi(s) = O(s - \xi)^2. \quad (72)$$

Let us write the Taylor series of  $T(s)$  and  $\phi(s)$  about  $\xi$ :

$$\begin{aligned} T(s) &= T(\xi) + (s - \xi)T'(\xi) + O(s - \xi)^2, \\ \phi(s) &= u_0 + u_1(s - \xi) + O(s - \xi)^2, \end{aligned}$$

for some  $u_1 \in \mathbb{C}^p$ . Then

$$T(s)\phi(s) = T(\xi)u_0 + (s - \xi)(T'(\xi)u_0 + T(\xi)u_1) + O(s - \xi)^2,$$

and we must have

$$\begin{aligned} T(\xi)u_0 &= 0 \\ T'(\xi)u_0 + T(\xi)u_1 &= 0. \end{aligned}$$

By symmetry the first equation implies  $u_0^T T(\xi) = 0$  and then the second one yields  $u_0^T T'(\xi)u_0 = 0$ . Since  $u_0 \neq 0$ , we obtain the desired result from Lemma 4 with  $u = u_0/\|u_0\|$ .

Assume next that the kernel of  $T(\xi)$  has dimension at least 2 over  $\mathbb{C}$ , and let  $\mathcal{V}$  be a 2-dimensional subspace. Two cases can occur:

(i) There exists a non-zero  $v \in \mathcal{V}$  such that  $T(s)v$  vanishes at  $\xi$  with order at least 2. On grouping the partial multiplicities in such a way that  $\sigma_j = 0$  for  $1 \leq j \leq j_0$ ,  $\sigma_j = 1$  for  $j_0 < j \leq j_1$ , and  $\sigma_j \geq 2$  for  $j_1 < j \leq p$ , we deduce that in the decomposition

$$F(s)v = \sum_{j=1}^{j_0} \phi_j(s)e_j + \sum_{j=j_0+1}^{j_1} \phi_j(s)e_j + \sum_{j=j_1+1}^p \phi_j(s)e_j$$

the functions  $\phi_j$  vanish at  $\xi$  with order at least 2 for  $1 \leq j \leq j_1$  and at least 1 for  $j_0 < j \leq j_1$ . In particular  $F'(\xi)v$  lies in the span of the  $e_j$  for  $j_0 < j \leq p$  which is the kernel of  $T(\xi)F^{-1}(\xi)$ . Therefore if we set  $\phi(s) := F^{-1}(s)F(\xi)v$ , we get an analytic function about  $\xi$  such that  $T(\xi)\phi(\xi)v = 0$  and

$$(T\phi)'(\xi) = T'(\xi)v - T(\xi)F^{-1}(\xi)F'(\xi)v = 0 + 0 = 0$$

so that (72) holds. The result now follows as in the previous part of the proof with  $u_0 = v$ .

(ii) Each non-zero  $v \in \mathcal{V}$  is such that  $T(s)v$  vanishes at  $\xi$  with order 1. By Takagi's factorization

$$U^T T(\xi) U = \text{diag}(0, \dots, 0, \rho_1, \rho_2, \dots, \rho_r),$$

for some unitary matrix  $U$ , where  $0 \leq r \leq p - 2$  and  $0 < \rho_1 \leq \rho_2 \leq \dots \rho_r$ . We may assume that the first two columns of  $U$  form a basis of  $\mathcal{V}$ . Define a matrix-valued analytic function about  $\xi$  by

$$Q(s) := U^T T(s) U = \text{diag}(0, \dots, 0, \rho_1, \rho_2, \dots, \rho_r) + (s - \xi)Q'(\xi) + O(s - \xi)^2.$$

The matrix  $Q'(\xi)$  is symmetric, and its  $2 \times 2$  left upper-block  $R_1(\xi)$  is invertible. Otherwise indeed, there would exist a non-zero  $y = [y_1, y_2, 0, \dots, 0]^T \in \mathbb{C}^p$  such that  $T(s)Uy$  vanishes at  $\xi$  with order 2, but this contradicts our standing assumption since  $Uy \in \mathcal{V}$  by the choice of  $U$ .

We now exhibit a non-zero vector  $w$  which satisfies the interpolation conditions

$$Q(\xi)w = 0 \tag{73}$$

$$w^T Q'(\xi)w = 0. \tag{74}$$

To meet (73), we put  $w = (x, 0, \dots, 0)^T$  with  $x \in \mathbb{C}^2$  so that  $w^T Q'(\xi)w = x^T R_1(\xi)x$ . By Lemma 3 we may write  $U_1^T R_1(\xi) U_1 = \Lambda$ , with  $U_1$  is a  $2 \times 2$  unitary matrix, where  $\Lambda = \text{diag}[\lambda_1^2, \lambda_2^2]$  is strictly positive. Define now  $v^T := \frac{1}{\sqrt{\lambda_1^2 + \lambda_2^2}} [\lambda_2 \quad i\lambda_1]$ . Obviously

$$v^T \Lambda v = \frac{1}{\lambda_1^2 + \lambda_2^2} [\lambda_2, -i\lambda_1] \begin{bmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{bmatrix} \begin{bmatrix} \lambda_2 \\ -i\lambda_1 \end{bmatrix} = 0,$$

hence setting  $x = U_1 v$  we have that  $w^T Q'(\xi)w = x^T R_1(\xi)x = v^T \Lambda v = 0$ , as desired. Finally, if we let  $u = U[w^T, 0]^T$ , we get  $u \in \mathcal{V}$  and putting  $B(s) := I_p + (b_\xi(s) - 1)uu^*$  we see from Lemma 4 that  $B^*(s)^T T(s) B^*(s)$  is analytic about  $\xi$ , which achieves the proof.  $\square$

**Theorem 2** *Let  $S$  be a  $p \times p$  symmetric Schur function of McMillan degree  $n$  which is strictly contractive at infinity. Let further  $H$  be the state characteristic matrix defined by (24) for some minimal symmetric realisation  $(A, B, C, D)$  of  $S$ . Assume that  $\chi_H$  has exactly  $\kappa$  distinct roots in  $\Pi^+$  with odd algebraic multiplicity. Then,  $S$  has a  $2p \times 2p$  symmetric inner extension of McMillan degree  $n + \kappa$ , and this extension has minimal degree among all such extensions of  $S$ .*

**Proof.** We apply Proposition 5 to the symmetric inner extension of Proposition 4:

$$\Sigma_{\check{P}} = \mathcal{S}_{\check{P}} \begin{bmatrix} \check{Q} & 0 \\ 0 & I_p \end{bmatrix}$$

in which  $\check{P}$  is the minimal solution to (31). Recall that  $\check{P}$  is the only solution to the Riccati equation such that  $\sigma(\hat{A} + \check{P}\hat{C}^*\hat{C}) \subset \bar{\Pi}^-$  (see section 4.2), or equivalently  $\sigma(\hat{A}^* + \hat{C}^*\hat{C}\check{P}) \subset \bar{\Pi}^+$ . By (36),

(39), and Proposition 4, the eigenvalues of  $H$  in  $\Pi^-$ , which are (counting multiplicity)  $n - n_0$  in number by (61), are precisely the poles of  $\check{Q}$ . Likewise the eigenvalues of  $H$  in  $\Pi^+$  are the zeros of  $\check{Q}$ , that are reflected from its poles across the imaginary axis with corresponding multiplicities. Let  $\xi_1 \in \Pi^+$  be one of these with multiplicity strictly greater than one. By proposition 5, there exists an inner function  $B = B_{\xi_1, u}$  such that  $T_1 := B^{-T} \Sigma_{\check{P}} B^{-1}$  is inner of degree  $2n - n_0 - 2$ . Clearly  $T_1$  is symmetric and moreover, since

$$\text{Ker} \begin{bmatrix} \check{Q}(\xi_1) & \\ & I_p \end{bmatrix} \subset \text{Ker} \Sigma_{\check{P}}(\xi_1),$$

it follows from the proposition that  $u$  may be chosen of the form  $[\tilde{u}^T, 0^T]^T$  (each block being of size  $p$ ) in which case  $B$  is of the form

$$\begin{bmatrix} B_0 & 0 \\ 0 & I_p \end{bmatrix},$$

so that  $T_1$  is an extension of  $S$ . The matrix  $H$  has  $n - n_0 = \kappa + 2\ell$  eigenvalues in  $\Pi^+$ , counting multiplicities. Thus, we can perform  $\ell$  iterations to obtain an extension of degree

$$2n - n_0 - 2\ell = n + \kappa.$$

We now prove that this extension has minimal McMillan degree. Let  $\Sigma$  be any symmetric extension of  $S$ . By Proposition 2, it can be written in the form

$$\Sigma = \begin{bmatrix} L & 0 \\ 0 & I_p \end{bmatrix} \mathcal{S}_P \begin{bmatrix} R & 0 \\ 0 & I_p \end{bmatrix} \quad \text{with} \quad \mathcal{S}_P = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix},$$

where  $L, R$ , are inner, and  $\mathcal{S}_P$  is an inner extension of  $S$  of the same degree. The extension  $\Sigma$  being symmetric, we must have

$$(LS_{12})^T = S_{21}R \Leftrightarrow S_{21}^{-1}S_{12}^T = R\bar{L}.$$

By Proposition 3, the degree of the unitary matrix  $S_{21}^{-1}S_{12}^T$  cannot be less than  $\kappa$ . This yields

$$\kappa \leq \deg S_{21}^{-1}S_{12}^T = \deg R\bar{L} \leq \deg R + \deg L,$$

so that

$$n + \kappa \leq n + \deg R + \deg L = \deg \Sigma.$$

□

**Corollary 2** *Let  $S$  be a symmetric Schur function of size  $p \times p$  which is strictly contractive at infinity. Then, the following propositions are equivalent*

- (i)  $S$  has a symmetric inner extension of size  $(2p) \times (2p)$  of the same McMillan degree,
- (ii) there is a Hermitian solution  $P$  to the algebraic Riccati equation (31) that satisfies  $PP^T = I_p$ ,
- (iii) the characteristic polynomial of the matrix  $H$  given by (21)-(24) can be written

$$\chi_H(s) = \pi(s)^2 \tag{75}$$

for some polynomial  $\pi$ .

Moreover, all extensions (i) are parameterized by (26), where  $U_1 = U_2^T$  and  $P$  satisfies (ii).

## 7 Completion of a real Schur function

We now assume that the Schur function  $S(s)$  is real that is to say

$$S(s) = \overline{S(\bar{s})}.$$

It thus admit a realization (20) in which  $A, B, C$  and  $D$  are real matrices. We have that

**Theorem 3** *Let  $P$  be a Hermitian solution to (31). Then, the associated minimal unitary extension  $S_P$  of  $S$  is real if and only if  $P$  is real.*

**Proof.** Formula (27) for  $S_P$  shows that if  $P$  is real, then  $S_P$  is real. Conversely, assume that  $S_P$  is real. Formula (27) shows that

$$C(sI_n - A)^{-1}PC^*$$

is real for real  $s$ . This implies that  $CA^kPC^*$  is real for  $k = 0, 1, \dots$ , and from the observability of  $(C, A)$  it follows that  $PC^*$  is real. The equation (31) can be rewritten in the form

$$\begin{aligned} & PC^*(I_p - DD^*)^{-1}CP + (A + BD^*(I_p - DD^*)^{-1}C)P \\ & + P(A^* + C^*(I_p - DD^*)^{-1}DB^*) + B(I_p - D^*D)^{-1}B^* = 0, \end{aligned}$$

and we see now that  $AP + PA^*$  is real. Write  $P = P_1 + iP_2$  where  $P_1$  and  $P_2$  are real matrices. Since  $CP$  is real, we have  $CP_2 = 0$  and  $P_2A^* + AP_2 = 0$ . Premultiply the later equation by  $C$  to obtain  $CAP_2 = 0$ , and by induction  $CA^kP_2 = 0$ ,  $k = 0, 1, \dots$ . Now, by the observability of  $(C, A)$ ,  $P_2 = 0$  and  $P$  is real.  $\square$

A real minimal realization of a symmetric function may fail to be symmetric. However there exist a minimal real realization which is signature symmetric (see [YT] or [F2, th.6.1]), that is to say

$$A^T = JAJ, \quad B^T = CJ, \quad C^T = JB, \quad D^T = D,$$

for some signature matrix

$$J = \begin{bmatrix} I_r & 0 \\ 0 & -I_{n-r} \end{bmatrix}.$$

It follows that  $\hat{A}^T = J\hat{A}J$ ,  $(\hat{B}\hat{B}^*)^T = J\hat{C}^*\hat{C}J$ ,  $(\hat{C}^*\hat{C})^T = J\hat{B}\hat{B}^*J$ , which implies that if  $P$  is a solution of (31), then  $\tilde{P} = JP^{-T}J$  is also a solution and we have  $S_{\tilde{P}} = S_P^T$ . If  $S_P$  is a real inner extension of  $S$  computed as in Theorem 1, then an analog of Proposition 3, in which  $P^{-T}$  must be replaced by  $\tilde{P}$ , allows to construct a real symmetric unitary extension of  $S$  with the same properties. However, the situation is more involved than in the complex case. Even if the algebraic multiplicity of the eigenvalues of  $H$  are all even, a symmetric real extension at the same degree may not exist. We conclude by illustrating this with an example.

### 7.1 An example

Consider the rational matrix

$$S(s) = \begin{bmatrix} f(s) & 0 \\ 0 & f(s) \end{bmatrix},$$

where  $f(s)$  is a strictly proper scalar Schur function of McMillan degree 1. Let

$$f(s) = d + c(s - a)^{-1}c,$$

be a symmetric realization of  $f(s)$ . The matrix-valued function  $S(s)$  is Schur and contractive at infinity and has the minimal realization

$$S(s) = D + C(sI_2 - A)^{-1}C^T, \quad \text{with} \quad D = dI_2, \quad C = cI_2, \quad A = aI_2.$$

In this example, all the eigenvalues of  $H$  defined in (24) have even algebraic multiplicity. We first construct a *complex* symmetric extension at the same degree, degree 2. The Riccati equation associated with  $S(s)$  is (see th.3.1)

$$\hat{C}^* \hat{C} P^2 + (\hat{A} + \hat{A}^*)P + \hat{B} \hat{B}^* = 0, \quad (76)$$

where

$$\hat{A} = \frac{a(1 - |d|^2) + c^2 \bar{d}}{1 - |d|^2} I_2, \quad \hat{B} \hat{B}^* = \frac{|c|^2}{1 - |d|^2} I_2, \quad \hat{C}^* \hat{C} = \frac{|c|^2}{1 - |d|^2} I_2.$$

The equation (76) can be rewritten as

$$P^2 + 2 \frac{\operatorname{Re}(\hat{A})(1 - |d|^2)}{|c|^2} P + I_2 = 0. \quad (77)$$

A Hermitian solution to (77) can be diagonalized in the form  $P = U \operatorname{diag}(\lambda, \mu) U^*$ , where  $U$  is an unitary matrix. Note that the eigenvalues of  $P$  must be solutions to the scalar Riccati equation

$$p^2 + 2 \frac{\operatorname{Re}(\hat{A})(1 - |d|^2)}{|c|^2} p + 1 = 0, \quad (78)$$

and must be positive real, since  $f(s)$  is Schur (positive real lemma). By corollary 2, a complex symmetric extension is obtained from a solution  $P$  to (77) which satisfies  $PP^T = I_2$ . Such a solution is obtained taking

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix},$$

and for  $\lambda$  and  $\mu$  the two distinct solutions to (78) which satisfy  $\lambda\mu = 1$ . The solution is

$$P = \frac{1}{2} \begin{bmatrix} \lambda + \mu & i(\lambda - \mu) \\ -i(\lambda - \mu) & \lambda + \mu \end{bmatrix}.$$

It is easy to check that  $P$  is Hermitian and moreover that  $PP^T = I_2$ .

We now come to the *real case* and assume that  $a, c$  and  $d$  are real. The Schur function  $S(s)$  has a real extension if and only if the Riccati equation (77) has a real solution  $P$ . Such a real solution must be of the form

$$P = O \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix} O^T,$$

where  $O$  is a real orthogonal matrix, and  $\lambda$  and  $\mu$  are (positive real) solutions to the scalar Riccati equation (78). Note that  $S(s)$  is Schur if and only if (78) has a positive real solution, which happens if and only if  $a \leq -\frac{c^2}{1-d}$ . A symmetric real extension of  $S(s)$  at the same degree is obtained from a real solution  $P$  to (77) which in addition satisfies  $PP^T = I_2$ , but this may only happen if  $P = I_2$ , that is when  $a = -c^2/(1-d)$ , in which case the two solutions of (78) are equal to 1. For example, the function  $f(s) = 1/(s + \zeta)$  is Schur if and only if  $\zeta \geq 1$ . It has a real symmetric extension of degree 2 if and only if  $\zeta = 1$ .

## References

- [A1] B.D.O. ANDERSON, Algebraic Properties of Minimal Degree Spectral Factors *Automatica* 9 (1973), 491-500.
- [AV] B. D. O. ANDERSON AND S. VONGPANITLERD, *Network Analysis and Synthesis - A modern systems theory approach*, Prentice Hall, New Jersey, 1973.
- [A2] D. Z. AROV, On the Darlington method in the theory of dissipative systems, *Dokl. Akad. Nauk. USSR* 201(3) (1971), 559-562.
- [A3] D. Z. AROV, Realization of matrix-valued functions according to Darlington, *Izv. Akad. Nauk SSSR Ser. Mat.* 37 (1973), 1299-1331.
- [BGR] J. A. BALL, I. GOHBERG AND L. RODMAN, *Interpolation of rational matrix functions*, Birkhäuser, 1990.
- [BEGO] L. BARATCHART, P. ENQVIST, A. GOMBANI, AND M. OLIVI Surface Acoustic Wave Filters, Unitary Extensions and Schur Analysis *Report 44 (2003)*, Mittag-Leffler Institute.
- [BO1] L. BARATCHART AND M. OLIVI, Inner-unstable factorization of stable rational transfer functions, in *Modeling, Estimation and Control of Systems with Uncertainty*, G.B. Di Masi and A. Gombani and A.B. Kurzhansky eds., *Progress in System and Control Theory* 10, 22–39, Birkhauser, 1991.
- [BO2] L. BARATCHART AND M. OLIVI, Critical points and error rank in best  $H^2$  matrix rational approximation of fixed McMillan degree, *Constructive Approximation* 14 (1998), 273–300.
- [BGK] H. BART, I. GOHBERG AND M. A. KAASHOEK, *Minimal factorization of matrix and operator functions*. Operator Theory: Advances and Applications 1, Birkhuser Verlag, Basel, 1979.
- [B1] V. BELEVITCH Elementary applications of the scattering formalism to network design, *IRE Trans. Circuit Theory* CT-3 (1956), 97-104.
- [BM] A. BULTHEEL AND K. MÜLLER., On several aspects of  $J$ -inner functions in Schur analysis, *Report TW Katholieke Universiteit Leuven*, 268, October 1997.
- [CN] F.M. CALLIER AND C.D. NAHUM, Necessary and Sufficient Conditions for the Complete Controllability and Observability of Systems in Series Using the Coprime Factorization of a Rational Matrix, *IEEE Trans.on Circuits and Systems* CAS-22(2) (1975), 90-95.
- [D1] S. DARLINGTON, Synthesis of reactance 4-poles which produce a prescribed insertion loss characteristics, *J. Math. Physics* 18 (1939), 257–355.
- [D2] P. DEWILDE, Roomy scattering matrix synthesis, *Tech. Rep. Dept. of Mathematics, Univ. of California, Berkeley*, 1971.
- [D3] P. DEWILDE, Generalized Darlington Synthesis, *IEEE Trans. on Circuits and Systems*, 45(1) (1999), 41–58.



- [DH] R. G. DOUGLAS AND J.W. HELTON, Inner dilations of analytic matrix functions and Darlington synthesis, *Acta Scientiarum Mathematicarum* 34 (1973), 61-67.
- [DSS] R. DOUGLAS, H. SHAPIRO, AND A. SHIELDS, Cyclic vectors and invariant subspaces for the backward shift operator, *Annales de l'Institut Fourier (Grenoble)* 20 (1970), 37-76.
- [D4] H. DYM,  $J$ -contractive matrix functions, reproducing kernel spaces and interpolation, *CBMS lecture notes* 71, American mathematical society, Rhodes island, 1989.
- [F1] P.A. FUHRMANN, *Linear Systems and Operators in Hilbert spaces*, McGraw-Hill, New York, 1981.
- [F2] P.A. FUHRMANN, On symmetric rational transfer functions, special issue on Linear Systems and Control, *Linear Algebra and Applications* 50 (1983), 167-250.
- [F3] P.A. FUHRMANN, On the characterization and parametrization of minimal spectral factors, *Journal of Mathematical Systems, Estimation and Control* 5 (1995), 383-444.
- [FH] P.A. FUHRMANN AND U. HELMKE, Complex parameterization of real transfer functions, *Archiv für Elektronik und Übertragungstechnik* 49 (1995), 293-306. Special issue in memory of Wilhelm Cauer.
- [G] J. B. GARNETT, *Bounded Analytic Functions*, Academic Press, 1981.
- [GLR] I. GOHBERG AND P. LANCASTER AND L. RODMAN, *Invariant subspaces of matrices with applications*, John Wiley, 1986.
- [GR] I. GOHBERG AND S. RUBINSTEIN, Proper contractions and their unitary minimal extensions, *Operator Theory: Advances and Applications* 33 (1988), 223-247.
- [HJ] R. A. HORN, C.R. JOHNSON *Matrix Analysis* Cambridge University Press, 1985.
- [KFA] R. E. KALMAN, P. L. FALB AND M. A. ARBIB, *Topics in Mathematical Systems Theory*. McGraw-Hill, 1969.
- [LR1] P. LANCASTER AND L. RODMAN, Solution of the Continuous and Discrete Time Algebraic Riccati Equations: A Review, in *The Riccati Equation*, chapter 2, 11-49, S. BITTANTI, A.J. LAUB, AND J.C. WILLEMS, EDITORS, Springer Verlag, 1991.
- [LR2] P. LANCASTER AND L. RODMAN, *Algebraic Riccati Equations*. Oxford University Press, 1995.
- [P] V.P. POTAPOV, The multiplicative structure of  $J$ -contractive matrix functions, *Amer. Math. Soc. Transl.* 15(2) (1960), 131-243.
- [W] J.C. WILLEMS, Least squares stationary optimal control and the algebraic Riccati equation, *IEEE Trans. Autom. Contr.* AC-16 (1971), 621-634.
- [YT] C. YOULA AND P. TISSI, N-port Synthesis via Reactance Extraction-Part. I, *IEEE. Inter. Convention. Record* (1966), 183-205.
- [YWP] D. C. YOULA, F. WINTER AND S. U. PILLAI, A New Study of the Problem of Compatible Impedances, *Int. J. of Circuits and Systems* 25 (1997), 541-560.